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Physical Information Entropy and Probability Shannon Entropy

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In a previous work by one of us (R. Urigu) concerning open quantum systems it was remarked that in processes of the type $w \to (\overline{p}_i \overline{w}_i)$, when evaluating the information entropy of the environment as the Shannon entropy of the outcome probabilities $\overline{p}_i$ in the channels $\overline{w}_i$, the total information entropy may decrease. We remark here that this decrease is easily excluded by requiring a condition of quantum modelizability of the environment even with respect to Shannon entropy ("cybernetic interpretability" of the environment). Further conditions on the quantum model of the environment are defined ("maximal observability" and "Boolean interpretability"), which are proved to be equivalent, and it turns out that, once satisfied in one model, they also are in any model with pure initial state; furthermore, these conditions turn out to be equivalent to the condition that the process consists of pure operations of the first kind. The relevance to the concept of macroscopicity and to the "von Neumann chain" is discussed.

GENERAL INTRODUCTION

The purpose of this work is to consider some aspects of the connection between macroscopic and quantum structure, especially between Shannon entropy and physical (information) entropy. The suggestion came from some questions which arose in our previous work (Urigu, 1989, 1993).

Such questions require some description of the environment of a quantum system. This is developed in Section 1; Section 1.5 collects the main results, especially involving "pure operations of the first kind": such results are a consequence of a very close connection between the operations which a quantum system undergoes and the concurrent modifications of the environment.

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Section 2 deals with information entropy, which is the main concern of the paper; to avoid repetitions the subject is introduced at the beginning of that section.

Discussions and conclusions are developed in the Sections 3 and 4.

1. DESCRIPTION OF THE ENVIRONMENT

1.1. External Processes

We start from the known treatment of operations, in particular as found in Kraus's lecture notes (Kraus, 1983), concerning the description of one operation (see also Hellwig and Kraus, 1969, 1970; Kraus, 1971).

We need some elementary adaptations and developments. In fact *for a given quantum system* we are, as in Ludwig (1961), Ascoli and Urigu (1984), Urigu (1989, 1993), and Ascoli (1993), interested in processes where an original ensemble w is split into a countable family of ensembles (or channels) $\overline{w_1}, \overline{w_2}, \ldots$:

$$w \longrightarrow (\overline{p_i}\overline{w_i})_{i \in I}$$

with $I = \{1, 2, \ldots\}$, conditional to unspecified effects on the environment occurring with probabilities $\overline{p_1}, \overline{p_2}, \ldots$ such that $\Sigma_i \overline{p_i} = 1$.

We may refer to them as "discrete" "scattering" processes, two complementary examples being any nondestructive measurement of an observable and the splitting of a beam by a partially reflecting mirror.

Actually, besides the effects conditioning the outgoing channels, there is physically also some effect conditioning the ingoing channel (the preparation). We prefer to put this latter effect on the same footing as the others: so we attach to it an extra-index 0 and we extend the index set I to $\Gamma = \{0\} \cup I$. So we consider a probability $\overline{p_0}$ also, which is equal to zero in the above-considered "scattering" processes, but may well be considered as different from 0, corresponding to a final ensemble $\overline{p_0}\overline{w_0}$, when considering "decay" processes.

So, in conclusion, we consider processes of the type $w \rightarrow (\overline{p_i}\overline{w_i})_{i \in \Gamma}$ (to simplify the notation we drop the 0 index in the initial channel). Hence we are interested in models describing "normalized" families $\mathcal{T} = (T_i)_{i \in \Gamma}$ of operations [discrete instruments in the terminology of Davies (1976)] rather than single operations.

Definition 1. Let a given quantum system be described in $(H, \mathcal{W})$, where H is the Hilbert space and $\mathcal{W}$ the set of the states. Let $\Gamma = \{0, 1, 2, \ldots\}$ (finite or countable). We say that the quantum system *undergoes a (discrete)* *external process,* more specifically *a (discrete) interaction with a (Γ-effected) environment,* whenever it undergoes a process

$$W \rightarrow (T_i(W) = \overline{p_i}^W \overline{W_i})_{i \in \Gamma}$$

described by a countable family $\mathcal{T} = (T_i)_{i \in \Gamma}$ of (completely positive) operations (Kraus, 1983) normalized in the sense that

$$\sum_{i \in \Gamma} \overline{p_i}^W = 1$$

where $W, \overline{W_i} \in \mathcal{W}$ and $\forall W \in \mathcal{W}$, $\overline{p_i}^W = \operatorname{tr} T_i(W)$ (whenever $\overline{p_i}^W = 0$ we mean that $\overline{W_i}$ may be chosen to be any state)

In particular, we speak of a *scattering process* whenever

$$\forall W \in \mathcal{W}, \qquad \overline{p_0}^W = 0 \qquad \text{equivalently} \qquad T_0(W) = 0$$

1.2. Phenomenological Description and Minimal External Boolean Model

As regards the environment, its simplest description, which we call a *phenomenological description,* concerns nothing but the probabilities $(\overline{p_i})_{i \in \Gamma}$ of the different effects conditioning the ingoing channel and the outgoing channels. Hence in this description the environment is characterized by nothing but the index set Γ (equivalently I), so that it is considered as a cybernetic rather than a physical reality; it is important for the following that this is the appropriate framework for the Shannon entropy.

Such a phenomenological description of the environment may be hypostatized by identifying the latter with a classical or macroscopic physical system; we prefer the term Boolean system. For instance, we may think of Γ as the set of the memory positions in a computer assisting the experiment.

Definition 2. For a given finite or countable set Γ we call a *Boolean system with phase space* Γ a system whose properties are described by the Boolean lattice L^b generated by Γ and whose set of the states $\mathcal{W}^b \subseteq [0, 1]^{\operatorname{card} \Gamma}$ consists of the probabilities $p = (p_i)_{i \in \Gamma}$ on Γ. We call δ_i the probability concentrated on $i \in \Gamma$, so that

$$\forall p \in \mathcal{W}^b, \qquad p = \sum_i p_i \delta_i$$

Definition 3. For a given quantum system described in $(H, \mathcal{W})$ undergoing a discrete external process described by $\mathcal{T} = (T_i)_{i \in \Gamma}$ (Definition 1), we call *minimal Boolean model of the environment,* equivalently *associated to* $\mathcal{T}$, the Boolean system (Definition 2) that has phase space Γ, which, $\forall W \in \mathcal{W}$, undergoes the process

$$\mathcal{W}^b \rightarrow \mathcal{W}^b\colon \quad p = \delta_0 \rightarrow \overline{p}^W = \sum_i \overline{p_i}^W \delta_i = \sum_i \operatorname{tr}(T_i(W))\delta_i$$

1.3. External Quantum Systems

If one is interested in more detail in the states of the environment, more complex models are required. We are not interested here in more complex Boolean models. The reference to Boolean (classical) systems may be shifted one step toward the von Neumann chain by requiring, in analogy to Hellwig and Kraus's description of one operation, that the environment be modelizable as an "external quantum system" according to the next definition. In this definition, of course, we pay attention to the final states of the external system, too, under the assumption that the measurement it undergoes is perfect.

Definition 4. For a given quantum system described in $(H, \mathcal{W})$ we speak of a *[finally (perfectly) measured] (Γ-effected) external quantum system,* with $\Gamma = \{0, 1, 2, \ldots\}$ referring to a model $(H', \mathbf{U}, \mathcal{P}')$ consisting of:

- A quantum system which is described in a Hilbert space H' and the set of whose states we call $\mathcal{W}'$.
- A unitary evolution $\mathbf{U}$ in the space $\mathbf{H} = H \otimes H'$ in which the composite system is described.
- A discrete decomposition of the identity $\mathcal{P}' = (P'_i)_{i\in\Gamma}$ in H'.

For a given initial state W' of the external system, we define the *induced (discrete) external process described by* $\mathcal{T} = (T_i)_{i\in\Gamma}$ as the external process that is conditioned by the family of effects of the external system described by $(P'_i)_{i\in\Gamma}$.

We are also interested in the family $\mathcal{T}' = (T'_i)_{i\in\Gamma}$ of operations from $\mathcal{W}$ to $\mathcal{W}'$ which leads to the final states of the external system after it has undergone a perfect nondestructive measurement of $\mathcal{P}' = (P'_i)_{i\in\Gamma}$.

That is, explicitly, with

$$\overline{\mathbf{W}} = \mathbf{U}(W \otimes W')\mathbf{U}^*, \qquad \forall i \in \Gamma \quad \mathbf{T}_i(W) = (I \otimes P'_i)\overline{\mathbf{W}}(I \otimes P'_i) \quad (1)$$

$\mathcal{T}$ and $\mathcal{T}'$ are expressed by (tr^0 and tr' being in $\mathbf{H}$ the trace operations with respect to H and H', respectively)

$$\forall i \in \Gamma \quad T_i\colon \quad W \to T_i(W) = \text{tr}'\, \mathbf{T}_i(W) = \text{tr}'((I \otimes P'_i)\overline{\mathbf{W}}) \qquad (2)$$

$$\forall i \in \Gamma \quad T'_i\colon \quad W \to T'_i(W) = \text{tr}^0\, \mathbf{T}_i(W) = P'_i(\text{tr}^0\, \overline{\mathbf{W}})P'_i \qquad (3)$$

so that finally the compound system undergoes the evolution

$$(W, W') \to ((T_i(W), T'_i(W)))_{i\in\Gamma} = ((\overline{p_i}^W \overline{W_i}, \overline{p_i}^W \overline{W'_i}))_{i\in\Gamma}$$

with $\overline{p_i}^W = \text{tr}\, T_i(W) = \text{tr}\, T'_i(W) = \text{tr}\, \mathbf{T}_i(W)$, $\Sigma_{i\in\Gamma}\, \overline{p_i}^W = 1$, where, whenever $\overline{p_i}^W = 0$, it is meant that $\overline{W_i}$ is chosen arbitrarily within $\mathcal{W}$ and $\overline{W'_i}$ within $\mathcal{W}'$ with $\overline{W'_i} \le P'_i$.

Clearly, the set of the final states of a finally measured external quantum system is included in the set of the states of a quantum system with superselection rules described in $(H'_i)_{i\in\Gamma}$, with $H'_i = \text{im } P'_i$. Actually something more may be said:

Proposition 1. For any finally measured external quantum system (Definition 4) the final states are naturally embedded in the set of the states of a quantum system with superselection rules described in $(K'_i = Q'_i H')_{i\in\Gamma}$, where

$$\forall i \in \Gamma \qquad K'_i = Q'_i H' = \overline{\sum_{W\in\mathcal{W}} \text{im } T'_i(W)} \subseteq H'_i = P'_i H'$$

the lattice whose properties we call L^e and the set whose states $\mathcal{W}^e \subseteq (\mathbf{R}_+\mathcal{W}'_i)_{i\in\Gamma}$, where $\mathcal{W}'_i$ denotes the set of the states on K'_i. More specifically, if L^b denotes the lattice of the minimal Boolean model associated to $\mathcal{T}$ (Definition 3), the mapping

$$i \in \Gamma \to K'_i = Q'_i H'$$

characterizes an embedding of L^b into L^e satisfying

$$L^b \to L^e \quad \text{is such that} \quad \forall W \in \mathcal{W} \quad \forall i \in \Gamma \quad \overline{p_i}^W = \text{tr } \mathbf{T}_i(W)$$
$$= \text{tr}((I \otimes P'_i)\mathbf{W})$$

Whenever, furthermore, $W' \leq Q'_0$, the process undergone by the external system is embedded into a process within $\mathcal{W}^e$:

$$\mathcal{W}^e \to \mathcal{W}^e\text{:} \quad W'\delta_0 = (\delta_{i0}W')_{i\in\Gamma} \to (T'_i(W))_{i\in\Gamma} = (\overline{p_i}^W \overline{W'_i})_{i\in\Gamma}$$

1.4. Quantum Models of the Environment of an External Process

According to Definition 4, any quantum model of the environment characterizes an external process described by a suitable family $\mathcal{T}$ of operations. The converse question, whether any given external process arises from a suitable quantum model of the environment, is answered by the equivalence of conditions 1 and 3 of the next proposition, which easily adapts to the problem considered here well-known results (Kraus, 1983) separately concerning the equivalence of conditions 1, 2 and 2, 3 in the case of one operation.

Proposition 2. For a given quantum system described in $(H, \mathcal{W})$, let $\mathcal{T} = (T_i)_{i\in\Gamma}$ be a family of positive linear mappings in the state space $\mathcal{V}$ of H (Davies, 1976). Then the following conditions are equivalent

1. $\mathcal{T} = (T_i)_{i\in\Gamma}$ describes an external process (Definition 1), i.e., is a "normalized" family of completely positive operators.

2. Any $T_i \in \mathcal{T}$ may be expressed by

$$T_i: \quad W \to T_i(W) = \sum_{\lambda \in N_i} A_\lambda W A_\lambda^* \tag{4}$$

where $\forall i \in \Gamma$, $(A_\lambda)_{\lambda \in N_i}$ is a countable family of bounded operators such that

$$\sum_{i \in \Gamma} \sum_{\lambda \in N_i} A_\lambda^* A_\lambda = 1$$

3. $\mathcal{T}$ allows an external (finally measured) quantum model (Definition 4).

The implications 3 ⇒ 2 (at least in part) and 2 ⇒ 3 are justified by the constructions implied in the next Propositions 3 and 4, respectively, which chiefly introduce in the most concise way the notations and the expressions required in this paper. Actually, Proposition 3 refers to the case in which the initial state W' of the external quantum system is pure with $W' \leq P_0'$: this case is the one we shall be mostly concerned with in the rest of the paper, as it corresponds to initial entropy $S' = 0$; the condition $W' \leq P_0'$ expresses the requirement that the initial state be conditioned by the effect described by P_0' and it does not amount to a restriction in what follows, due to Remark 2 before Definition 6.

Proposition 3. Let $(H', \mathbf{U}, \mathcal{P}')$ describe a finally measured external quantum system according to Definition 4.

Let us consider and label in H' an orthonormal basis $(e_\lambda')_{\lambda \in N}$ in accord with the following scheme (here λ is a "two-figures" index, $N = \cup_{i \in \Gamma} N_i$):

spans	$\overbrace{\hphantom{e_{01}', e_{02}', \ldots}}^{P_0'H'}$	$\overbrace{\hphantom{e_{11}', e_{12}', \ldots}}^{P_1'H'}$	$\overbrace{\hphantom{e_{21}', e_{22}', \ldots}}^{P_2'H'}$	$\cdots$	(a)
basis vectors	$e_{01}', e_{02}', \ldots$	$e_{11}', e_{12}', \ldots$	$e_{21}', e_{22}', \ldots$	$\cdots$	(b)
index subsets	$\underbrace{\hphantom{e_{01}', e_{02}', \ldots}}_{N_0}$	$\underbrace{\hphantom{e_{11}', e_{12}', \ldots}}_{N_1}$	$\underbrace{\hphantom{e_{21}', e_{22}', \ldots}}_{N_2}$	$\cdots$	(c)

Let the operators $\mathbf{H} = H \otimes H'$ be represented by means of matrices of operators in H according to the decomposition

$$\mathbf{H} = \bigoplus_{\lambda \in N} (H \otimes e_\lambda')$$

Then, in the case of an external quantum system with pure initial state such that $W' = (W')^2 \leq P_0'$, let us choose e_{01}' so that

$$|e_{01}'\rangle\langle e_{01}'| = W'$$

and let us call $(A_\lambda) = (\mathbf{U}_{\lambda 0})$ the first column of the matrix $(\mathbf{U}_{\lambda\mu})$ representing $\mathbf{U}$:

$$\forall \lambda = N \quad A_\lambda = \mathbf{U}_{\lambda 0} \tag{5}$$

Then the operator $\mathbf{W} = \mathbf{U}(W \otimes W')\mathbf{U}^*$ [see (1)] is found to be represented by the matrix

$$(\overline{\mathbf{W}}_{\lambda\mu}) = (\mathbf{U}_{\lambda 0} W \mathbf{U}^*_{\mu 0}) = (A_\lambda W A^*_\mu) \tag{6}$$

Hence [see (1)–(3)]

$$\forall i \in \Gamma \quad \forall \lambda, \mu \in N_i \qquad (\mathbf{T}_i(W))_{\lambda\mu} = ((I \otimes P'_i)\overline{\mathbf{W}}(I \otimes P'_i))_{\lambda\mu}$$
$$= \overline{\mathbf{W}}_{\lambda\mu} = A_\lambda W A^*_\mu \tag{7}$$

$$\forall i \in \Gamma \qquad T_i(W) = \mathrm{tr}'((I \otimes P'_i)\overline{\mathbf{W}})$$
$$= \sum_{\lambda \in N_i} A_\lambda W A^*_\lambda \tag{8}$$

$$\forall i \in \Gamma \quad \forall \lambda, \mu \in N_i \qquad (T'_i(W))_{\lambda\mu} = \mathrm{tr}(\overline{W}_{\lambda\mu})$$
$$= \mathrm{tr}(A_\lambda W A^*_\mu) = \mathrm{tr}(A^*_\mu A_\lambda W) \tag{9}$$

The above expressions for the $\mathbf{T}_i$, T_i, and T'_i arise through an elementary calculation, due to the matrix representation of the operators that is described above (see work by Hellwig and Kraus cited above), and in particular the second one may be used as a step in the justification of the implication 3 ⇒ 2 of Proposition 2.

In the next, converse proposition, as well as in the subsequent Theorem 1, we explicitly emphasize the important case in which the following Condition 1 on $\mathcal{T} = (T_i)_{i \in \Gamma}$ is satisfied, because then some statements are simplified.

Condition 1. The external process (Definition 1) described by the family of operations $\mathcal{T} = (T_i)_{i \in \Gamma}$ undergone by a given quantum system described in $(H, \mathcal{W})$:

- is a scattering process ($T_0 = 0$)
- or the Hilbert space H is finite-dimensional
- or, more generally, the T_i may be expressed by $T_i(W) = \Sigma_{i \in N_i} A_\lambda W A^*_\lambda$ [see (4)], where at least one of the A_λ, say A_{01} (we are using "two-figure" indices), allows a von Neumann polar decomposition $A_{01} = U_{01}\sqrt{A^*_{01}A_{01}}$ with unitary U_{01}.

We may now state the converse statement of Proposition 3.

Proposition 4. Conversely, let a given quantum system described in $(H, \mathcal{W})$ undergo an external process described by $\mathcal{T} = (T_i)_{i \in \Gamma}$, with $\Gamma = \{0, 1, 2, \ldots\}$ (Definition 1). Let each T_i be expressed as in condition 2 of Proposition 2 by $T_i(W) = \Sigma_{\lambda \in N_i} A_\lambda W A^*_\lambda$, the index subsets N_i partitioning an index set N, as displayed in line (c) of the scheme of Proposition 3 (λ being a "two-figure" index).

• If the simplifying Condition 1 is satisfied, let us take a Hilbert space H' spanned by an orthonormal basis $(e'_\lambda)_{\lambda \in N}$ indexed by $\lambda \in N$ [line (b) in the above scheme] and in H' let us choose $P'_0, P'_1, P'_2, \ldots$ according to line (a) of the scheme and put $\mathcal{P}' = (P'_i)_{i \in \Gamma}$. Let us call $\mathcal{A}$ the column matrix constructed with the operators $A_{02}, A_{03}, \ldots, A_{11}, A_{12}, A_{13}, \ldots$ (we are using "two-figure" indexes) and let U_{01} be the unitary operator entering the polar decomposition of A_{01} (see Condition 1); let us define [in analogy to the construction of Hellwig and Kraus (1969) concerning two "complementary" operations; see Urigu (1989)], the operator U in $\mathbf{H} = H \otimes H'$ (which is known to be unitary)

$$\mathbf{U} = \begin{pmatrix} A_{01} & \dfrac{U_{01}\mathcal{A}^*}{\sqrt{I_{\mathbf{H}\ominus H} - \mathcal{A}\mathcal{A}^*}} \\ \mathcal{A} & \end{pmatrix} \tag{10}$$

Let us choose $W' = |e'_{01}\rangle\langle e'_{01}|$.

• In general, that is, even when Condition 1 is not satisfied, let us [in analogy to Kraus (1983)] add in the above construction of H' an extra dimension with basis vector e'_{00} to be included within P'_0H' (so that $00 \in N_0$); let us include A_{01}, too, in the above definition of $\mathcal{A}$ and define on $\mathbf{H} = H \otimes H'$

$$\mathbf{U} = \begin{pmatrix} 0_H & \mathcal{A}^* \\ \mathcal{A} & I_{\mathbf{H}\ominus H} - \mathcal{A}\mathcal{A}^* \end{pmatrix} \tag{11}$$

Let us choose $W' = |e'_{00}\rangle\langle e'_{00}|$.

Then these definitions complete in any case the characterization of a finally measured external quantum system $(H', \mathbf{U}, \mathcal{P}')$ which, with the pure initial state W', just induces the given family $\mathcal{T}$ of operations as expressed by formula (8) of Proposition 3.

We remark that even when $\mathcal{T}$ does not satisfy Condition 1 and $i = 0$, formulas (7)–(9) of Proposition 3 still hold because the additional term due to the extra dimension is easily seen to be 0_H by extending (6) to the extra dimension and using the fact that the first element of the matrix (11) is 0_H.

1.5. Environment of "First-Kind External Processes"

The main results of Section 1 of this paper are arranged within the next Definition 5–7 and the next Theorem 1.

In the three definitions, concerning a given external process described by $\mathcal{T} = (T_i)_{i \in \Gamma}$, three possible properties of $\mathcal{T}$ (we also speak of properties of the environment of the process) are defined through three corresponding requirements on the finally measured quantum models of $\mathcal{T}$ (Definition 4 and Proposition 3), such definitions being justified because (a) Proposition

4 ensures that the appropriate quantum models of $\mathcal{T}$ exist, and (b) the proof after Theorem 1 ensures that, whenever a quantum model of $\mathcal{T}$ satisfying any one of the three requirements exists, then all the quantum models of $\mathcal{T}$ with pure initial state W' satisfy the same requirement.

Definition 5. Let a given quantum system described in $(H, \mathcal{W})$ undergo an external interaction $W \to (T_i(W))_{i\in\Gamma}$, $\Gamma = \{0, 1, 2, \ldots\}$ with a Γ-effected environment (Definition 1).

We say that *(in the process) the environment satisfies maximal observability* whenever equivalently (see the proof after Theorem 1):

1a. In some quantum model of $\mathcal{T} = (T_i)_{i\in\Gamma}$ with pure initial state W' (Definition 4) the final states $(\mathbf{T}_i(W))_{i\in\Gamma}$ of the compound system [formula (1)], which are conditioned by the effects described by the orthogonal projectors $(P'_i)_{i\in\Gamma}$, may also be conditioned by the effects described by one-dimensional orthogonal projectors $Q'_i \leq P'_i$, that is, with $\overline{\mathbf{W}} = \mathbf{U}(W \otimes W')\mathbf{U}^*$,

$\forall i \in \Gamma \quad \forall W \in \mathcal{W}$

$$\begin{aligned}\mathbf{T}_i(W) &= (I \otimes P'_i)\overline{\mathbf{W}}(I \otimes P'_i) = (I \otimes Q'_i)\overline{\mathbf{W}}(I \otimes Q'_i)\\ &= \mathrm{tr}'((I \otimes Q'_i)\overline{\mathbf{W}}) \otimes Q'_i = T_i(W) \otimes Q'_i \qquad (12)\end{aligned}$$

1b. In all quantum models of $\mathcal{T}$ with pure initial state W' the same condition holds.

Remark 1. It turns out from the proof after Theorem 1 that, if the Q'_i satisfying the above condition exist, then they coincide with those defined in Proposition 1.

The formulation of the next definition implies the following remark:

Remark 2. For a given external process described by $\mathcal{T}$, let $(H', \mathbf{U}, \mathcal{P}')$ with pure initial state W' be a quantum model of $\mathcal{T}$ (Definition 4). Then, for a given one-dimensional projector Q'_0 in H', by modifying $\mathbf{U}$ only, we obtain a new external quantum system $(H', \tilde{\mathbf{U}}, \mathcal{P}')$ such that, with the initial state $\tilde{W}' = Q'_0$, it induces the same families of operations $\mathcal{T}$ and $\mathcal{T}'$.

In fact, if U' is any unitary operator in H' such that $W' = U'\tilde{W}'U'^*$, by simply taking $\tilde{\mathbf{U}} = \mathbf{U}(1 \otimes \tilde{U}')$, one obtains in the new model the same final state $\mathbf{W}$ [formula (1)], hence the same families $\mathcal{T}$ and $\mathcal{T}'$.

Definition 6. Let a given quantum system described in $(H, \mathcal{W})$ undergo an external interaction $W \to (T_i(W))_{i\in\Gamma}$, $\Gamma = \{0, 1, 2, \ldots\}$ with a Γ-effected environment (Definition 1).

We say that *(in the process) the environment satisfies Boolean interpretability* (or *macroscopic* or more improperly *classical* interpretability) whenever equivalently (see the proof after Theorem 1):

2a. In some quantum model $(H', \mathbf{U}, \mathcal{P}')$ of $\mathcal{T} = (T_i)_{i\in\Gamma}$ (Definition 4), with reference to Proposition 1, the subspaces (Q_i') of H' introduced there are one-dimensional, that is,

$$\forall i \in \Gamma \quad \forall W \in \mathcal{W} \qquad T_i'(W) = \bar{p}_i{}^W Q_i' \tag{13}$$

(namely, the T_i' are degenerate pure operations), equivalently the embedding $L^b \to L^e$ induced by the mapping $i \in \Gamma \to K_i' = Q_i'H'$ (see Proposition 1) may be reduced to an identification $L^b = L^e$; we require $W' = Q_0'$, too.

2b. In all quantum models $(H', \mathbf{U}, \mathcal{P}')$ of $\mathcal{T}$ with pure initial state W', after a suitable adjustment of the unitary operator $\mathbf{U}$ (see Remark 2) the same condition holds.

Remark 3. Concerning condition 2a of Boolean interpretability, we remark that the identification $L^b = L^e$ which is required also implements an identification $\mathcal{W}^b = \mathcal{W}^e$: more precisely, $(H', \tilde{\mathbf{U}}, \mathcal{P}')$ with the initial state $\tilde{W}' = Q_0'$ (see Remark 2) supplies a quantum model of the environment in the strong sense that the mapping $i \to K_i' = Q_i'H'$ provides an identification j of $L^b \cup \mathcal{W}^b$ and $L^e \cup \mathcal{W}^e$ satisfying

$$j\colon\quad L^b \cup \mathcal{W}^b \xrightarrow{j} L^e \cup \mathcal{W}^e \text{ is such that}$$

$$\begin{array}{lccc} \text{the Boolean process} & p = \delta_0 & \to & \bar{p}^W \\ \text{is identified with} & \downarrow j & & \downarrow j \\ \text{the quantum process} & W' = Q_0' & \to & (T_i'(W)) \end{array}$$

Definition 7. Let a given quantum system described in $(H, \mathcal{W})$ undergo an external interaction $W \to (T_i(W))_{i\in\Gamma}$, $\Gamma = \{0, 1, 2, \ldots\}$ with a Γ-effected environment (Definition 1).

We say that *(in the process) the environment satisfies cybernetic interpretability* or *the Shannon entropy has a physical meaning* (compare with Proposition 6 of that subsection 2.3) whenever equivalently (see the proof after Theorem 1):

3a. In some quantum model of $\mathcal{T} = (T_i)_{i\in\Gamma}$ with initial entropy $S' = 0$, equivalently with pure initial state W' (Definition 4), for any initial state W of the given system the final states $T_i'(W)$ of the external system in all the channels are pure.

3b. In all quantum models of $\mathcal{T}$ with initial entropy $S' = 0$, equivalently with pure initial state W', the same condition holds.

We also use the following terminology:

Definition 8. Let a given quantum system described in $(H, \mathcal{W})$ undergo an external process described by $\mathcal{T} = (T_i)_{i \in \Gamma}$ with $\Gamma = \{0, 1, 2, \ldots\}$. We speak of a *process of the first kind* whenever each T_i is pure of the first kind, that is it may be expressed by

$$\forall i \in \Gamma \qquad T_i\colon \quad W \to T_i(W) = A_i W A_i^* \qquad (A_i \text{ bounded operators})$$

Theorem 1. Let a given quantum system undergo an external process (Definition 1). Then, with reference to Definitions 5–8, the following conditions are equivalent:

0. The external process is of the first kind.
1. In the process the environment satisfies maximal observability.
2. In the process the environment satisfies Boolean interpretability.
3. In the process the environment satisfies cybernetic interpretability or the Shannon entropy has a physical meaning.

Actually, whenever these conditions are satisfied, there exists a quantum model of the environment $(H', \mathbf{U}, \mathcal{P}')$, $\mathcal{P}' = (P'_i)_{i \in \Gamma}$ (Definition 4) with a pure initial state W' such that (a) if the external process satisfies Condition 1 (before Proposition 4), then the P''s are one-dimensional, and (b) in general they also are one-dimensional, except for P'_0, being two-dimensional.

An Example. Before proceeding to the required proofs, let us show a simple example. For the given trivial process whose simplest model has $\mathcal{P}' = \{P'_0\}$ with one-dimensional P'_0, a model with two-dimensional P'_0 is exhibited, so as to explicitly verify that the projector Q'_0 on $\Sigma_{W \in \mathcal{W}}$ im $T'_0(W)$ (see Proposition 1) still remains one-dimensional: hence $Q'_0 \neq P'_0$.

Let us refer to the trivial one-channel decay process, where $\Gamma = \{0\}$ and $\mathcal{T}$ consists of the identical operation $W \to T_0(W) = W$ only. Then, in the simplest model, constructed according to the prescription of Proposition 3, one has

$$H' = \mathbf{C}, \qquad \mathbf{U} = 1_{H \otimes \mathbf{C}}, \qquad \mathcal{P}' = \{P'_0\} = \{1\},$$
$$W' = P'_0 = 1, \qquad \overline{\mathbf{W}} = W \otimes 1, \qquad T'_0(W) = 1$$

So in this model, with reference to Proposition 1, one has $K'_0 = \Sigma_{W \in \mathcal{W}}$ im $T'_0(W) = Q'_0 H' = P'_0 H' = H'$.

Let us construct a model of the same trivial process where this equality does not hold. Let us choose

$$H' = \mathbf{C}^2 \text{ with the canonical basis } (e'_{01}, e'_{02})$$
$$\mathbf{U} = 1 \otimes \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}, \qquad \beta_1, \beta_2 \in \mathbf{C}, \qquad |\beta_1|^2 + |\beta_2|^2 = 1$$
$$\mathcal{P}' = \{P'_0\} = \{1_{H'}\}, \qquad W' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then

$$\mathbf{T}_0(W) = (I \otimes P_0')\overline{\mathbf{W}}(I \otimes P_0') = \mathbf{W} = W \otimes \begin{pmatrix} \beta_1\beta_1^* & \beta_1\beta_2^* \\ \beta_2\beta_1^* & \beta_2\beta_2^* \end{pmatrix} \quad (14)$$

$$= W \otimes Q_0' = (I \otimes Q_0')\overline{\mathbf{W}}(I \otimes Q_0') \quad (15)$$

where Q_0' turns out to be the orthogonal one-dimensional projector on $\beta_1 e_{01}' + \beta_2 e_{02}'$. Hence

$$T_0(W) = \mathrm{tr}'(\mathbf{T}_0(W)) = W, \qquad T_0'(W) = \mathrm{tr}^0(\mathbf{T}_0(W)) = Q_0'$$

Thus, with reference to Proposition 1, we can see that in this model $K_0' = \Sigma_{W \in \mathcal{W}} \text{ im } T_0'(W) = Q_0'H' \neq P_0'H' = H'$, so that conditions 1a, 2a, 3a of Definitions 5, 6, and 7 are satisfied with $Q_0' \neq P_0'$. Moreover, with reference to Remark 2 and Definition 6, the unitary operator $\mathbf{U}$ may be adjusted to $\tilde{\mathbf{U}} = 1_{H \otimes H'}$ so as to obtain a new quantum model such that with the initial state $\tilde{W}' = Q_0'$ the same operations T_0 and T_0' are induced.

The example may be physically interpreted as describing a beam of light hitting a splitter whose outputs are afterward rejoined without observation (e.g., Michelson interferometer); were the two outputs to be rejoined after an observation, then the given operation T_0 would be changed, specifically to a nonpure operation).

Proof. We prove here the equivalence of the seven conditions 0 of Theorem 1, 1a, 1b of Definition 5, 2a, 2b of Definition 6, and 3a, 3b of Definition 7, and we also prove the last statement of Theorem 1.

We first prove the equivalence of 1a, 2a, 3a, which concern the existence of quantum models of $\mathcal{T}$ with definite properties. We use the fact that Remark 2 allows to satisfy requirement $W' = Q_0'$ in 2a in any case.

• It is easily seen that 1a $\Rightarrow$ 2a. In fact, according to 1a of Definition 5,

$$\forall i \in \Gamma \quad \forall W \in \mathcal{W} \quad \mathbf{T}_i(W) = (I \otimes Q_i')\overline{\mathbf{W}}(I \otimes Q_i') \quad \text{with 1D } Q_i' \leq P_i'$$

Taking the partial trace with respect to H [as from (1) to (3)], we immediately obtain

$$T_i'(W) = \mathrm{tr}^0\, \mathbf{T}_i(W) = \mathrm{tr}^0(T_i(W) \otimes Q_i') = \overline{p_i}^W Q_i'$$

so that 2a of Definition 6 is satisfied.

• Let us now compare conditions 2a and 3a. We see that 2a $\Rightarrow$ 3a as a particular case: in fact 2a of Definition 6 may be expressed as stating that for any channel i the final states $T_i'(W)$ of the environment, which both conditions 2a and 3a (of Definition 7) require to be pure, actually depend on the initial state W of the given system only through the numerical factors $\overline{p_i}^W$.

• But we may see that 3a $\Rightarrow$ 2a, too, that is, if 3a holds, then for any i the one-dimensional subspaces im $T'_i(W)$ do not depend on W: in fact, as the mapping from $\mathcal{W}$ into $\mathcal{W}'$: $W \to T'_i(W)$ is linear, if for some channel i we had $W_1 \to T'_i(W_1) \neq 0$ and $W_2 \to T'_i(W_2) \neq 0$ with im $T'_i(W_1) \neq$ im $T'_i(W_2)$, then any proper mixture W of W_1 and W_2 would be mapped into a proper mixture [of the pure states $T'_i(W_1)$ and $T'_i(W_2)$], contrary to condition 3a, which has been assumed.

Let us now prove that 2a $\Rightarrow$ 1a. Let $(H', \mathbf{U}, \mathcal{P}')$ with a given pure initial state W' (Definition 4) be a quantum model of the external system satisfying condition 2a of Definition 6. Let us refer to the construction in Proposition 3: as $\forall i \in \Gamma$, $Q'_i \leq P'_i$ (see Proposition 1), we may choose in H' the orthonormal basis $(e'_\lambda)_{\lambda \in N}$ introduced there so that

$$\text{with} \quad e'_{\lambda_i} = e'_{i1} \qquad \forall i \in \Gamma \quad \mathbf{C}e'_{\lambda_i} = \text{im } Q'_i$$

Then formula (13) of Definition 6 may be expressed as

$$\forall i \in \Gamma \quad \forall \lambda, \mu \in N_i \quad \forall W \in \mathcal{W} \qquad (T'_i(W))_{\lambda\mu} = \delta_{\lambda\mu}\delta_{\lambda\lambda_i}\overline{p_i}^W \tag{16}$$

Let us now refer to the corresponding representation of $\mathbf{U}$ as a matrix $(\mathbf{U}_{\lambda\mu})$ of operators of H, with the notation $A_\lambda = \mathbf{U}_{\lambda 0}$ for its first column, and to formula (9) of Proposition 3. Then formula (16) with $\mu = \lambda$ becomes

$$\forall i \in \Gamma \quad \forall \lambda \in N_i \quad \forall W \in \mathcal{W} \qquad (T'_i(W))_{\lambda\lambda} = \text{tr}(A^*_\lambda A_\lambda W) = \delta_{\lambda\lambda_i}\overline{p_i}^W$$

hence

$$\forall i \in \Gamma \quad \forall \lambda \in N_i \qquad A_\lambda = \delta_{\lambda\lambda_i}A_\lambda$$

So, according to formula (7) of Proposition 3,

$$\forall i \in \Gamma \quad \forall \lambda, \mu \in N_i \quad \forall W \in \mathcal{W}$$

$$\begin{aligned}(\mathbf{T}_i(W))_{\lambda\mu} &= ((I \otimes P'_i)\overline{\mathbf{W}}(I \otimes P'_i))_{\lambda\mu} \\ &= \overline{\mathbf{W}}_{\lambda\mu} = A_\lambda W A^*_\mu = \delta_{\lambda\lambda_i}\delta_{\mu\lambda_i}A_\lambda W A^*_\mu = \delta_{\lambda\lambda_i}\delta_{\mu\lambda_i}\overline{\mathbf{W}}_{\lambda\mu} \\ &= ((I \otimes Q'_i)\overline{\mathbf{W}}(I \otimes Q'_i))_{\lambda\mu}\end{aligned}$$

so that formula (12) of Definition 5 referring to maximal observability, that is, condition 1a of that definition, is satisfied.

• We may also use the last formula to conclude that 2a (hence even 1a or 3a) $\Rightarrow$ 0, external process of the first kind: in fact, for any $i \in \Gamma$, by taking the partial trace tr$'$, we get [see formula (8) of Proposition 3]

$$T_i(W) = \sum_{\lambda \in N_i} A_\lambda W A^*_\lambda = A_{\lambda_i} W A^*_{\lambda_i}$$

which clearly represents an operation of the first kind.

• We may now easily see that, conversely, condition 0 of the theorem, external process of the first kind, implies the existence of quantum models satisfying the condition of maximal observability 1a of Definition 5 and even the last statement of the theorem. In fact if $\mathcal{T}$ satisfies condition 0 of the theorem, the index subsets $(N_i)_{i\in\Gamma}$, as introduced in condition 2 of Proposition 2, may trivially be chosen to be one-element subsets. Then the explicit construction of Proposition 4 leads to an initially and finally measured quantum model of $\mathcal{T}$ such that (a) if $\mathcal{T}$ satisfies the simplifying Condition 1, then all the projectors P'_i are one-dimensional, and (b) in general they also are, except for P'_0, projecting on the two-dimensional subspace of H' spanned by the basis vectors e'_{00} and e'_{01}; still the corresponding final state $\mathbf{T}_0(W)$ of the compound system [see formula (1)] is conditioned also by the one-dimensional orthogonal projector $Q'_0 < P'_0$ on e'_{01}: in fact, due to the extension of (7) to the "extra-dimension" with basis vector e'_{00} and to (11)

$$(\mathbf{T}_0(W)|_{H\otimes P'_0H'}) = ((I\otimes P'_0)\overline{\mathbf{W}}(I\otimes P'_0)|_{H\otimes P'_0H'}) = \begin{pmatrix} 0 & 0 \\ 0 & A_0WA_0^* \end{pmatrix}$$

$$= ((I\otimes Q'_0)\overline{\mathbf{W}}(I\otimes Q'_0)|_{H\otimes P'_0H'})$$

So in any case, under condition 0 of the theorem, the model constructed according to Proposition 4 is seen to satisfy condition 1a of Definition 5 (hence conditions 2a and 3a we have already proved to be equivalent) and even the last statement of the theorem.

• What appears less evident is that, once for a given external process a quantum model with the properties specified in 1a, 2a, 3a of Definitions 5, 6, and 7 is assumed to exist (equivalently, whenever the process is of the first kind), then every quantum model with pure W' of the process satisfies the same properties, that is, conditions 1b, 2b, 3b hold. With reference to this, let us prove the implication $0 \Rightarrow$ 1b. The main tool is the remarkable Lemma of Hellwig and Kraus (1969), which leads to the explicit form and classification of the (completely continuous) pure operations.

Let $(H', \mathbf{U}, \mathcal{P}')$ with a pure initial state W' describe any quantum model inducing a pure first-kind process $\mathcal{T}$. Then, for any initial state W of the given system, let us refer to the representation introduced in Proposition 3 to express its final states $T_i(W)$; let in particular $W = |f\rangle\langle f|$, with $f \in H$ be any pure state: then we get, from formula (8) there,

$$T_i\colon\quad W = |f\rangle\langle f| \to T_i(W) = \sum_{\lambda\in N_i} A_\lambda W A_\lambda^* = \sum_{\lambda\in N_i} |A_\lambda f\rangle\langle A_\lambda f|$$

As T_i is a pure operation, $T_i(W)$ has to be a multiple of a one-dimensional projector in H. Then the extremality of the one-dimensional projectors within the cone of the positive linear operators leads us to state that

$$\forall f \in H \quad \forall \lambda \in N_i \qquad A_\lambda f = \alpha_\lambda(f) g(f), \qquad \alpha_\lambda(f) \in \mathbf{C}$$

where g is an operator in H independent on λ. At this point let us express the Hellwig–Kraus Lemma to be used, complemented with its connession with pure operations.

Lemma 1 (Hellwig–Kraus). Let (A_λ) be a family of nonzero linear operators of a Hilbert space H such that

$$\forall f \in H \quad \forall \lambda \qquad A_\lambda f = \alpha_\lambda(f) g(f), \qquad \alpha_\lambda(f) \in \mathbf{C}$$

where g is an operator in H independent of λ. Then one of the following alternatives holds:

(i) $\forall \lambda$, $A_\lambda = \alpha_\lambda B$, with $\alpha_\lambda \in \mathbf{C}$ and B linear operator in H, equivalently the operation T: $W \to T(W) = \Sigma_\lambda A_\lambda W A_\lambda^* = AWA^*$ [with $a = \alpha B$, $\alpha = \sqrt{\Sigma_\lambda |\alpha_\lambda|^2}$, so that $A_\lambda = (\alpha_\lambda/\alpha)A$] is (by definition) of the first kind.

(ii) $\forall \lambda$, $A_\lambda = |g\rangle\langle f_\lambda|$ with g, $f_\lambda \in H$ and at least two f_λ's linearly independent, equivalently the operation T: $W \to T(W) = \Sigma_\lambda A_\lambda W A_\lambda^* = \mathrm{tr}(WD)|g\rangle\langle g|$ with D a positive, bounded, linear operator in H, is (by definition) of the second kind.

The statements about $T(W)$ of course also require the sums to be convergent.

As in our case the operations are supposed to be of the first kind, alternative (i) occurs: then, for any fixed $i \in \Gamma$, by calling A_i the operator A of the lemma, we obtain from formulas (7) and (8) of Proposition 3,

$$\begin{aligned}
\forall \lambda, \mu \in N_i \qquad (\mathbf{T}_i(W))_{\lambda\mu} &= ((I \otimes P_i')\overline{\mathbf{W}}(I \otimes P_i'))_{\lambda\mu} = A_\lambda W A_\mu^* \\
&= A_i W A_i^* \frac{1}{\alpha^2} \alpha_\lambda \alpha_\mu^* \\
&= \left(A_i W A_i^* \otimes \left| \frac{1}{\alpha} \sum_{\nu \in N_i} \alpha_\nu e_\nu' \right\rangle \left\langle \frac{1}{\alpha} \sum_{\nu \in N_i} \alpha_\nu e_\nu' \right| \right)_{\lambda\mu} \\
&= (T_i(W) \otimes Q_i')_{\lambda\mu}
\end{aligned}$$

so that, according to formula (12) of Definition 5, the operation $\mathbf{T}_i$ which is conditioned by the projector P_i' is conditioned by the one-dimensional projector Q_i' on $\Sigma_{\nu \in N_i} \alpha_\nu e_\nu'$, too: hence the arbitrary quantum model with pure W' inducing $\mathcal{T}$ which has been considered satisfies formula (12) of Definition 5 expressing the condition of maximal observability.

Then, as argued in the first steps of the proof, the model even satisfies the conditions of Boolean interpretability and cybernetic interpretability, so that the deduction of the equivalence of the seven conditions displayed at the beginning of the proof is completed.

2. CYBERNETIC INTERPRETABILITY OF THE ENVIRONMENT

2.1. Introduction

Let us identify, at first without discussion, the information entropy of any given quantum system with the quantum-theoretic von Neumann entropy

$$S(W) = \operatorname{tr} s(W) \quad \text{with} \quad s(x) = -x \lg x$$
$$\text{(extended continuously to } x = 0) \qquad (17)$$

The grounds of this identification are analyzed in some detail in the discussion of Section 3.1.

Referring as in Section 1.1 to processes where, for a given quantum system, an original ensemble w splits into channels w_i, we consider as *final entropy* $\bar{S}(W)$ of the given system the average entropy of the channels or conditional entropy

$$\bar{S}(W) = \overline{S^c}(W) = \sum_{i\in\Gamma} \bar{p}_i \bar{S}_i(W) \equiv \sum_{i\in\Gamma} \bar{p}_i S(W_i)$$

In a previous work (Ascoli and Urigu, 1984) we derived a tendency to decrease for the entropy of the given system in pure discrete external processes (processes transforming pure states into pure states in each channel):

$$\text{for pure } \mathcal{T} \qquad \overline{S^c} \leq S \qquad (18)$$

(and one result of the present work is to clarify the meaning of the limitation to pure external processes and hopefully to reduce its weight).

Then the natural question arises of what becomes of total entropy, including the environment. We already remarked in Section 1.2 that, concerning the environment, the *phenomenological description* of the process concerns nothing but the final probabilities $\bar{p}_i$ of the unspecified external effects conditioning the different channels, so that the entropy concept that may be applied to the environment is the *Shannon entropy* of the probability distribution $(\bar{p}_i)$ (or "mixing entropy" of the channels)

$$\overline{S^b} = \sum_{i\in\Gamma} s(\bar{p}_i)$$

(its initial value S^b being considered 0, as initially there is one channel). This is a cybernetic rather than a physical concept and the environment, when endowed with this entropy, appears to be described cybernetically rather than physically, even after introduction of the minimal Boolean model of Definition 3. So we adopt the next definition.

Definition 9. Let a given quantum system undergo an interaction

$$W \to (T_i(W) = \overline{p_i}^W \overline{W_i})_{i \in \Gamma}$$

with a Γ-effected environment, $\Gamma = \{0, 1, 2, \ldots\}$ (Definition 1). Then we correspondingly consider [with $s(x)$ as in (17) above]:

• The transformation of the *(information) entropy of the given system*

$$S(W) = \operatorname{tr} s(W) \to \overline{S}(W) = \overline{S^c}(W) = \sum_{i \in \Gamma} \overline{p_i}^W S(\overline{W_i}) \tag{19}$$

• The transformation of the *Shannon entropy* (mixing or macroscopic entropy) *of the environment* (equivalently *associated to* $\mathcal{T}$)

$$S^b = \sum_{i \in \Gamma} s(p_i) = \sum_{i \in \Gamma} s(\delta_{0i}) = 0 \to \overline{S^b}(W) = \sum_{i \in \Gamma} s(\overline{p_i}^W) \tag{20}$$

Urigu (1989) compared the initial entropy S with the possible evaluation of the total final entropy as $\overline{S^c} + \overline{S^b}$: it was found that (see Definition 8)

$$\text{for a first kind } \mathcal{T}\ \overline{S^c} \leq S \leq \overline{S^c} + \overline{S^b} \tag{21}$$

the right inequality being not guaranteed for a general pure $\mathcal{T}$. Actually a simple counterexample has been given, which has been the main spur to this research. With $H = \mathbf{C}^2$, a one-channel external process $\mathcal{T} = (T)$ is considered, so that $\overline{S^b} = 0$; furthermore, T may be chosen to be a "pure operation of the second kind"

$$T\colon \quad W \to T(W) = P, \qquad P \text{ a 1D orthogonal projector} \tag{22}$$

The example may be made concrete as in the account of Urigu (1993), from which we show Fig. 1: an unpolarized beam of photons is decomposed, by a Nicol prism, according to two mutually orthogonal directions of linear polarization; then a 90° rotation of the polarization direction is performed on one of the outgoing beams; the re-collection of the beams, disregarding their spatial degrees of freedom, which do not fall within the model, gives

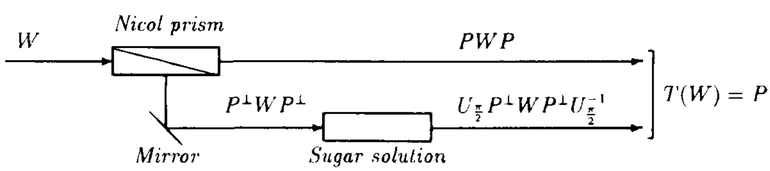

Fig. 1. A device which decreases the polarization entropy of light.

rise to a single linearly polarized channel. So unpolarized light is transformed into an equal amount of polarized light and we have

$$\lg 2 = S > \overline{S^c} + \overline{S^b} = 0$$

which contradicts the right inequality in (21).

Thus this example induces us to submit to criticism the identification of the final entropy of the environment with the Shannon entropy $\overline{S^b}$ of the outcome probability distribution $(\overline{p_i})$, equivalently with the physical entropy of the minimal Boolean model of the environment (Definition 3).

The guiding criterion we follow to solve the problem is the general idea of the *universality of quantum theory,* which amounts here to requiring quantum modelizability of the environment even with respect to information entropy as identified with Shannon entropy. Clearly it turns out that, whenever such a modelization is possible, the evolution laws of quantum theory ensure the tendency to increase of the total information entropy, as outlined in the next subsection.

2.2. Information Entropy in the Quantum Description of the Environment

Definition 10. Let a given quantum system undergo an interaction with an external finally measured quantum system (Definition 4)

$$(W, W') \to ((\overline{p_i}^W \overline{W_i}, \overline{p_i}^W \overline{W'_i}))_{i \in \Gamma}$$

Then we correspondingly consider:

• The transformation of the *(information) entropy of the given system* as before

$$S(W) = \operatorname{tr} s(W) \to \overline{S}(W) = \overline{S^c}(W) = \sum_{i \in \Gamma} \overline{p_i}^W S(\overline{W_i}) \tag{23}$$

• The transformation of the *(information) entropy of the external quantum system*

$$S' = S(W') \to \overline{S'}(W) = \overline{S'^c}(W) + \overline{S^b}(W) = \sum_{i \in \Gamma} \overline{p_i}^W S(\overline{W'_i}) + \sum_{i \in \Gamma} s(\overline{p_i}^W)$$

With these assignments of entropies, the tendency to increase of the total information entropy, as guaranteed by the evolution laws of quantum theory, may be expressed as follows.

Proposition 5. Under the assumptions of Definition 10, the total information entropy tends to increase:

$$\begin{aligned} \forall W \in \mathcal{W} \qquad S(W) + S' &\le \overline{S^c}(W) + \overline{S'}(W) \\ &= \overline{S^c}(W) + \overline{S'^c}(W) + \overline{S^b}(W) \end{aligned} \tag{25}$$

We express the straightforward proof (see also Urigu, 1993) reporting at each step in the upper line the transformation of the states and in the lower line the corresponding inequality for entropy; we have (due to entropy additivity and to entropy invariance in unitary evolutions)

$$\begin{cases} (W, W') \to \mathbf{W} = W \otimes W' \to \overline{\mathbf{W}} = U\mathbf{W}U^{-1} \to \\ (S, S') \to \mathbf{S} = S + S' = \overline{\mathbf{S}} \leq \end{cases}$$

(due to entropy increase associated to perfect non-destructive measurement)

$$\begin{cases} \to (\mathbf{T}_i(W) = (I \otimes P'_i)\overline{\mathbf{W}}(I \otimes P'_i) = \overline{p}_i \overline{\mathbf{W}}_i)_{i\in\Gamma} \to \\ \leq \overline{S^c} + \overline{S^b} = \sum_{i\in\Gamma} \overline{p}_i S(\overline{\mathbf{W}}_i) + \sum_{i\in\Gamma} s(\overline{p}_i) \leq \end{cases}$$

(due to entropy increase associated to decomposition of the composite system)

$$\begin{cases} \to ((T_i(W) = \mathrm{tr}'\, \mathbf{T}_i(W), T'_i(W) = \mathrm{tr}^0\, \mathbf{T}_i(W)))_{i\in\Gamma} \\ \leq \sum_{i\in\Gamma} \overline{p}_i S(\overline{W}_i) + \sum_{i\in\Gamma} \overline{p}_i S(\overline{W'_i}) + \sum_{i\in\Gamma} s(\overline{p}_i) = \overline{S^c} + \overline{S'^c} + \overline{S^b} \end{cases}$$

2.3. Information Entropy in the Interaction with a "Cybernetically Interpretable" Environment

Let us now compare (20) with (24). We see that quantum modelizability of the environment even with respect to information entropy as identified with Shannon entropy requires that the environment allows a quantum model such that its initial state W' and, for any initial state W of the given system, its final states in all the channels be pure. These are just the requirements of condition 3 of Theorem 1 (see Definition 7) as stated in the next proposition, which in this way justifies the terminology introduced there and exhibits the cybernetic relevance of Theorem 1.

Proposition 6. Let a given quantum system described in $(H, \mathcal{W})$ undergo an external interaction $W \to (T_i(W))_{i\in\Gamma}$, $\Gamma = \{0, 1, 2, \ldots\}$ with a Γ-effected environment (Definition 1). Then the condition of Definition 7 that *the environment satisfies cybernetic interpretability* or *the Shannon entropy has a physical meaning,* hence the equivalent conditions of Theorem 1, are satisfied if and only if the (finally measured) quantum models of the environment with pure initial state also modelize information entropy as identified with Shannon entropy, in the sense that initially and finally the information entropy of the quantum model coincides with the Shannon entropy of the environment; explicitly,

$$S' = S^b = 0 \tag{26}$$

$$\overline{S'} = \overline{S^b} = \sum_{i\in\Gamma} s(\overline{p}_i) \qquad \text{(that is, } \overline{S'^c} = 0) \tag{27}$$

Actually, according to the proof of Theorem 1, whenever the conditions (26) and (27) are satisfied in some quantum model of $\mathcal{T}$, they are satisfied in all the quantum models of $\mathcal{T}$ that satisfy the first one of them.

Thus, in particular, we see that only operations of the first kind satisfy the requirement of cybernetic interpretability of the environment we suggested as underlying the total information entropy growth as expressed by the right inequality (21) (Urigu, 1989). Of course, within the treatment we have developed here, this latter inequality immediately follows from Proposition 6, using Proposition 5 in the particular case $\overline{S'^c} = 0$ [whereas the left inequality (21) arises from the main result of Ascoli and Urigu (1984), which applies whenever the T_i are pure operations, not necessarily of the first kind].

In the next proposition we explicitly collect these conclusions concerning the dynamics of information entropy in the external processes we are considering. The equivalent conditions of Theorem 1, to which reference is made, suggest a further clarification of the subject, as outlined in the discussion which follows.

Corollary 1. Let a given quantum system undergo an external interaction

$$W \rightarrow (T_i(W) = \overline{p}_i{}^W \overline{W}_i)_{i \in \Gamma}$$

$\Gamma = \{0, 1, 2, \ldots\}$ with a Γ-effected environment (Definition 1), such that any one of the equivalent conditions of Theorem 1 be satisfied. Then, with reference to Definition 9;

- The information entropy of the given quantum system tends to decrease.
- The total information entropy, including the entropy of the environment identified as the Shannon entropy of $(\overline{p}_i{}^W)_{i \in \Gamma}$, tends to increase:

$$\overline{S^c} \leq S \leq \overline{S^c} + \overline{S^b}$$

This means

$$\sum_{i \in \Gamma} \overline{p}_i{}^W S(\overline{W}_i) \leq S(W) \leq \sum_{i \in \Gamma} \overline{p}_i{}^W S(\overline{W}_i) + \sum_{i \in \Gamma} s(\overline{p}_i{}^W) \tag{28}$$

3. DISCUSSION

3.1. Shannon Entropy and Physical Information Entropy

The treatment that has been developed here may lead to some insight into the relationship between probability Shannon entropy and physical information entropy.

In fact, as already remarked, Shannon entropy is a cybernetic concept, as it does not refer to any physical system, but to a probability distribution over a countable set Γ ("classical" probability).

When going over to physical reality there must be some point where for certain physical systems it has been decided that the information entropy of the system is defined as the Shannon entropy of some probability distribution associated with the system. The simplest physical system appears to be a "classical" discrete system with phase space Γ, the Boolean system of Definition 2: its physical information entropy S^b is identified with the Shannon entropy of the outcome probability of a maximal observation on the system: $S^b = \Sigma_{i\in\Gamma}\, s(\overline{p}_i)$.

To go over to quantum physics, we have to remember that quantum probabilities concern macroscopic effects on the environment (Ludwig, 1961, 1983). So there is no way out from referring to processes $W \to (T_i(W) = \overline{p}_i\overline{W}_i)_{i\in\Gamma}$ as introduced at the beginning of the paper, then considering the Shannon entropy $\overline{S^b}(W) = \Sigma_{i\in\Gamma}\, s(\overline{p}_i)$, and finally inducing the physical information entropy of the system from this Shannon entropy, which we may always interpret as the physical entropy of the minimal Boolean model associated with $\mathcal{T} = (T_i)_{i\in\Gamma}$ (Definition 3).

This requires recognizing processes $W \to (T_i(W) = \overline{p}_i\overline{W}_i)_{i\in\Gamma}$, which may be assumed as reversible, so that, referring to Definition 9, information entropy may be interpreted as transferred from the given system (entropy decrease $S - \overline{S^c}$) to the environment (entropy increase $\overline{S^b} - S^b$) without entropy production: that is, $S + S^b = \overline{S^c} + \overline{S^b}$.

A known process of this type is the perfect nondestructive measurement associated with a decomposition of the identity $\mathcal{P} = (P)_{i\in\Gamma}$, provided that it is applied to a system in a state $W = \Sigma_i\, P_iWP_i$ diagonal with respect to $\mathcal{P}$: it may be considered a reversible process because remixing of the channels $(P_iWP_i)_{i\in\Gamma}$ restores the original state W. The situation simplifies further if the initial Shannon entropy of the environment and the final information entropy of the given system vanish: $S^b = 0$, $\overline{S^c} = 0$. Then the identity $S = \overline{S^b}$ holds. The condition $S^b = 0$ is satisfied in any perfect nondestructive measurement, as initially the Shannon entropy of the environment, the measuring instrument, is assumed to be 0; the condition $\overline{S^c} = 0$ is satisfied, too, whenever the observation is maximal, that is, the P_i are one-dimensional: then the final states $\overline{W}_i$ in the channels are pure, so that the final entropy of the given system (conditional entropy as in Definition 9) is 0: $\Sigma_{i\in\Gamma}\, \overline{p}_iS(\overline{W}_i) = 0$.

So in a maximal, perfect, nondestructive measurement it is reasonable, and it is usually implicitly done, to identify the initial physical information entropy $S(W)$ of the given system with the final Shannon entropy $\overline{S^b}(W)$ of the environment. From this identification the usual von Neumann expression for the entropy of a given quantum system is easily obtained in the known way: for a given state W of the system, choose $\mathcal{P} = \mathcal{P}_W$, where $\mathcal{P}_W$ is a maximal decomposition of the identity diagonalizing W; then the above identification leads to

$$S(W) = \sum_i s(\overline{p}_i) = \sum_i s(\mathrm{tr}(P_i W)) = \sum_i s(\mathrm{tr}(P_i W P_i)) = \mathrm{tr}\, s(W)$$

Moreover, we remark here that this identification turns out to be consistent with the possibility of considering succeeding environments. In fact, when performing again on the measuring instrument a maximal, perfect, nondestructive measurement of its outcome effects, the same probability distribution $(\overline{p}_i)$ occurs: so this probability distribution and its associated Shannon entropy no longer depend on the number and physical realization of the subsequent perfect measurements which may be done, that is, they become cybernetic rather than physical concepts ("von Neumann chain").

3.2. Information Entropy in the Interaction with an External Quantum System

Of course, the counterexample considered in Section 3.1 shows that, when considering as before processes of the type $W \to (T_i(W) = \overline{p}_i \overline{W}_i)_{i \in \Gamma}$, the inequality [see Definition 10 and formula (28)] $S \leq \overline{S^c} + \overline{S^b}$ may be violated, so that evaluation of the information entropy of the environment as Shannon entropy of the probability distribution $(\overline{p}_i)_{i \in \Gamma}$ leads in general to unwanted results.

The main problem that has been considered here is in which cases, besides the perfect nondestructive measurement, such an evaluation of the entropy of the environment is allowed. As already stated at the end of Section 2.1, the general idea that has been followed is universality of quantum theory, which amounts here to requiring quantum modelizability of the environment with reference not only to the family $\mathcal{T}$ of operations, but even to the Shannon entropy of the probability distribution $(\overline{p}_i)_{i \in \Gamma}$; this means $\overline{S'} = \overline{S^b}$ (see Proposition 6): we have called this condition cybernetic interpretability of the environment or condition for the Shannon entropy to have a physical meaning. If this condition is satisfied, counterexamples like the one considered before are certainly excluded.

In fact, *when considering a quantum model of the environment,* due to the quantum laws of evolution, information entropy satisfies the inequality (25): $S + S' \leq \overline{S^c} + \overline{S'} = \overline{S^c} + \overline{S'^c} + \overline{S^b}$. Concerning the initial entropy $S + S'$, the processes considered here (Definition 1), according to Proposition 4, always allow a quantum model with a pure initial state W' of the environment, so that the initial entropy S' of the environment vanishes and (25) reduces to

$$S \leq \overline{S^c} + \overline{S'} = \overline{S^c} + \overline{S'^c} + \overline{S^b}$$

Then it is clear that no guarantee for the preservation of the inequality may be given if the term $\overline{S'^c}(W) = \Sigma_{i \in \Gamma} \overline{p_i}^W S(\overline{W'_i})$, the final conditional entropy of the environment, is dropped, as it is when stating $S \leq \overline{S^c} + \overline{S^b}$: the device of Fig. 1 shows that actually that term cannot in general be dropped. We see that the condition that it may be dropped, that is, $\overline{S'^c} = 0$, or $\overline{S'} = \overline{S^b}$, is just equivalent to the condition of cybernetic interpretability of the environment introduced above (see Proposition 6), which therefore is a sufficient condition to guarantee the validity of the inequality $S \leq \overline{S^c} + \overline{S^b}$, expressing the tendency to increase of the total information entropy, the entropy of the environment being evaluated as Shannon entropy.

The main result, Theorem 1, consists in necessary and sufficient conditions for such a cybernetic interpretability of the environment to hold. (We actually saw that something less expected occurs: whenever a quantum model with vanishing initial entropy satisfying these conditions exists, then all the quantum models with vanishing initial entropy satisfy them as well.)

3.3. Information Entropy and Environment

The most interesting condition for the above-considered cybernetic interpretability of the environment is perhaps that all the operations of the family $\mathcal{T} = (T_i)_{i \in \Gamma}$ be of the first kind [that is, expressed by $T_i(W) = A_i W A_i^*$, with A_i bounded operators]. As the operation T [formula (22)] performed by the device of Fig. 1 is not of the first kind, it certainly does not satisfy this condition, so that from the point of view of a quantum description of the environment the inequality $S \leq \overline{S^c} + \overline{S^b}$ may well be violated, as it is [a more detailed account of this counterexample may be found in Urigu (1993)].

Here we recognized the condition $\overline{S'} = \overline{S^b}$ that the Shannon entropy has a physical meaning as the underlying sufficient physical requirement for the inequality $S \leq \overline{S^c} + \overline{S^b}$ to hold and we found that operations of the first kind only satisfy this requirement.

Actually, as already stated in Urigu (1989), the condition that the interaction consists of operations of the first kind also ensures a tendency to decrease of the information entropy of the given system [see formula (18) above], as a consequence of the main statement of our previous work (Ascoli and Urigu, 1984), where the required condition was that the operations of the process be pure, as is certainly true for operations of the first kind; so the condition that the operations be of the first kind implies the double inequality $\overline{S^c} \leq S \leq \overline{S^c} + \overline{S^b}$ [see (21) and (28)].

A further insight into the subject may come from the equivalent condition 2 in Theorem 1, which refers to the possibility of constructing a quantum model of the environment whose initial and final states can be identified with the states of the minimal Boolean model introduced in Definition 3.

Actually, according to Proposition 1, any final state in a quantum model of the environment may be identified with a state of a quantum system with superselection rules, specifically with the family

$$(\overline{p}_i{}^W\overline{W'_i})_{i\in\Gamma} = (T'_i(W))_{i\in\Gamma} \tag{29}$$

of quantum states on a Γ-indexed family $(K'_i = Q'_iH')_{i\in\Gamma}$ of Hilbert spaces.

In general the structure of each state $\overline{W'_i}$ cannot be neglected, that is, the description of the external process through the family $\mathcal{T} = (T_i)_{i\in\Gamma}$ of operations only, equivalently the description of the environment through its Boolean model (Definition 3) only, does not provide enough information on the environment. In fact, when considering the information entropy $\overline{S'} = \overline{S'^c} + \overline{S^b}$ of the environment, it is just the structure of each state $\overline{W'_i}$ in (29) which is taken into account by the term $\overline{S'^c}$ whose vanishing expresses the condition that the Shannon entropy has a physical meaning (condition 3 of the same Theorem 1).

This point of view is further supported by condition 1 in Theorem 1, which requires the observation conditioning the channels in the quantum model describing the environment to be maximal.

In conclusion, referring again to condition 0 of Theorem 1 requiring the operations to be of the first kind, we may say that operations which are not of the first kind correspond to an unadmissibly poor description of the environment. When the description is adequate, the double inequality $\overline{S^c} \leq S \leq \overline{S^c} + \overline{S^b}$ [see (21) and (28)] is guaranteed. According to Theorem 1, the description may be considered as adequate if and only if the operations are of the first kind: this condition meets with the original intuition of Haag and Kastler (1961) concerning the very concept of operation.

4. CONCLUSIONS

4.1. *Semi-Boolean (Semimacroscopic)* and *Boolean (Macroscopic)* Environments in Given External Processes

Let us first look at the terminology. When applying the term "Boolean" to the environment in a given external process, we refer to the lattice of its propositions in the description that is considered; we avoid [as does Ludwig (1983)] the term "classical," which of course is appropriate from the "kinematical" point of view, because a quantum system is just identified through the nonclassical dynamical behavior of its effects on a Booleanly described environment; we intend the term Boolean to be equivalent to the term "macroscopic" (which could improperly suggest something necessarily big) which Ludwig uses in opposition to the term "microscopic," which he reserves for something requiring a quantum description. So, according to the considerations that have been developed in this work, it appears that, when considering a given quantum system undergoing a process of the type

$$W \rightarrow (T_i(W) = \overline{p_i}\,{}^{W}\overline{W_i})_{i \in \Gamma}$$

the environment, where the effects conditioning the channels take place, cannot in general be regarded as Boolean (macroscopic), or its information entropy be identical with the Shannon entropy of the probability distribution $(\overline{p_i})_{i \in \Gamma}$ of the effects: we may call it *semi-Boolean (semimacroscopic).*

However, according to the nature of the quantum model [which generalizes the description of one operation by Hellwig and Kraus (1969; Kraus, 1983)], the environment always has to be considered in turn as subject to a perfect nondestructive measurement by some further environment which may now be regarded as truly *Boolean (macroscopic)* because a perfect nondestructive measurement consists of operations of the first kind: so, according to Theorem 1 and Proposition 6, the information entropy of this latter environment may be identified with the Shannon entropy of the probability distribution $(\overline{p_i})_{i \in \Gamma}$.

From this point on, any further environment would remain Boolean (macroscopic) with the same information entropy $S^b = \Sigma_{i \in \Gamma}\, s(\overline{p_i})$, because the subsequent interactions may be considered as perfect nondestructive measurements: we would have the von Neumann chain; we could also speak of a *cybernetic chain.* So, when considering an interaction which is not of the first kind, one has to complete the von Neumann chain of succeeding environments with a first linking semi-Boolean (semimacroscopic) environment which cannot yet be described by a Boolean (macroscopic) model, but requires a quantum model.

4.2. An Interpretation from the Point of View of the Foundations of Quantum Physics

From the point of view of quantum interpretation, the situation may be described as follows.

Physical reality is grasped through macroscopic effects, hence for quantum systems through effects on the environment.

Whenever effects take place, some physical possibilities turn into actual events. The increase of total information entropy measures such a transition, which corresponds to a transition from quantum structure, which describes the physical possibilities, to Boolean (macroscopic) structure, which describes the actual events.

Thus in a process of the type $W \rightarrow (T_i(W) = \overline{p_i}\,{}^{W}\overline{W_i})_{i \in \Gamma}$ some possibilities are generally transformed into actual events, but, whenever $\mathcal{T} = (T)_{i \in \Gamma}$ is not of the first kind, according to Proposition 6 (see also Section 3.3), the environment generally still keeps some untransformed possibilities, so that it may be regarded as "semi-Boolean" ("semimacroscopic"), but not yet as Boolean (macroscopic). Only after it has interacted with a suitable subsequent environment may there remain no possibility which can turn into actuality, and this new environment may be regarded as Boolean (macroscopic); from that point on, information entropy remains constant and the further chain may be called cybernetic (see also Ascoli, 1993).

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